INEQUALITIES

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Contents

Chapter 1

MEANS INEQUALITIES

Definition 1 *Arithmetic mean of* $a_1, a_2, ..., a_n$ *is* $AM = \frac{a_1+a_2+...+a_n}{n}$

Definition 2 *Geometric mean of* $a_1, a_2, ..., a_n$ *is* $GM = \sqrt[n]{a_1 \cdot a_2 \cdot ... \cdot a_n}$

Definition 3 *Harmonic mean of* $a_1, a_2, ..., a_n$ *is* $HM = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}}$

Definition 4 *Quadratic mean of* $a_1, a_2, ..., a_n$ *is* $QM = \sqrt{\frac{a_1^2+a_2^2+...+a_n^2}{n}}$

Definition 5 *The rth power mean of* $a_1, a_2, ..., a_n$ *is* $P_r = \sqrt[r]{\frac{a_1^r+a_2^r+...+a_n^r}{n}}$

Theorem 1 *Let* $a_i \in \mathbb{R}_+$

$QM(a_1, a_2, ..., a_n) \geq AM(a_1, a_2, ..., a_n) \geq GM(a_1, a_2, ..., a_n) \geq HM(a_1, a_2, ..., a_n)$

and equality holds if and only if $a_1 = a_2 = ... = a_n$.

Theorem 2 *Let* $a_i \in \mathbb{R}_+$ *then* $P_{r_1} \geq P_{r_2}$ *whenever* $r_1 \geq r_2$.

Example 1 *Let* $a, b, c > 0$, *prove that* $a^2 + b^2 + c^2 \geq ab + bc + ca$.

Solution: By AM-GM inequality we have $a^2 + b^2 \geq 2ab$. Similarly, $b^2 + c^2 \geq 2bc$ and $a^2 + c^2 \geq 2ac$. Adding these three inequalities we get

$$2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca) \Longrightarrow a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Let's link this result since we will use it in many problems.

$$a^2 + b^2 + c^2 \geq ab + bc + ca \tag{1.1}$$

Example 2 *Prove that if* $a, b, c > 0$*, then* $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$.

Solution: By AM-GM we have $\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3} \geq \sqrt[3]{\frac{a\cdot b\cdot c}{b\cdot c\cdot a}} = 1$. So, $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$.

Example 3 *Prove that for any positive real numbers* a, b, c *we have*

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{9}{2(a+b+c)}.$$

Solution: By AM-HM inequality, we have

$$\frac{\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}}{3} \geq \frac{3}{\frac{1}{\frac{1}{a+b}} + \frac{1}{\frac{1}{b+c}} + \frac{1}{\frac{1}{c+a}}} = \frac{3}{2(a+b+c)}.$$

Therefore,

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{9}{2(a+b+c)}.$$

Example 4 *Prove that if* $a, b, c > 0$ *then*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}. \tag{1.2}$$

Solution: By the previous example we have,

$$\begin{aligned}
(a+b+c)[\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}] &\geq \frac{9}{2} \\
\Longleftrightarrow \frac{a+b+c}{a+b} + \frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} &\geq \frac{9}{2} \\
\Longleftrightarrow \frac{a}{b+c} + 1 + \frac{a}{b+c} + 1 + \frac{a}{b+c} + 1 &\geq \frac{9}{2} \\
\Longleftrightarrow \frac{a}{b+c} + \frac{a}{b+c} + \frac{a}{b+c} &\geq \frac{3}{2}.
\end{aligned}$$

This inequality is called Nesbitt's inequality.

Example 5 *Prove that* $\sqrt[3]{3+\sqrt[3]{3}} + \sqrt[3]{3-\sqrt[3]{3}} < 2\sqrt[3]{3}$.

Solution: By $AM - P_3$ we have,

$$\frac{\sqrt[3]{3+\sqrt[3]{3}} + \sqrt[3]{3-\sqrt[3]{3}}}{2} \leq \sqrt[3]{\frac{(\sqrt[3]{3+\sqrt[3]{3}})^3 + (\sqrt[3]{3-\sqrt[3]{3}})^3}{2}} = \sqrt[3]{3}.$$

Since the terms are not equal we have strict inequality.
So, $\sqrt[3]{3+\sqrt[3]{3}} + \sqrt[3]{3-\sqrt[3]{3}} < 2\sqrt[3]{3}$.

1.1 EXERCISES

1. Prove that for any positive real numbers a, b, c we have
$$(a+2)(b+3)(c+6) \geq 48\sqrt{abc}.$$

2. Prove that if $a, b, c > 0$, then $(a+b)(b+c)(c+a) \geq 8abc$.

3. Prove that if $a, b, c > 0$, then
$$\frac{ab}{c^2} + \frac{bc}{a^2} + \frac{ca}{b^2} \geq 3.$$

4. Prove that if $a, b, c > 0$, then
$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq a + b + c.$$

5. Prove that if $x, y > 0$ then,
$$\frac{x}{x^4 + y^2} + \frac{y}{x^2 + y^4} \leq \frac{1}{xy}.$$

6. Prove that $(a+b-c)(b+c-a)(c+a-b) \leq abc$ if
   (a) a, b, c are sides of a triangle
   (b) a, b, c are positive real numbers.

7. Prove that if $a, b, c > 0$ and $a^2 + b^2 + c^2 = 3$, then
$$\frac{1}{1+ab} + \frac{1}{1+bc} + \frac{1}{1+ca} \geq \frac{3}{2}.$$

8. Prove that if x, y, z are real numbers with $z > 0$, then
$$\frac{x^2 + y^2 + 12z^2 + 1}{4z} \geq x + y + 1.$$

9. Prove that the inequality $(3a+b+c)^2 \geq 12a(b+c)$ holds for any real numbers a, b, c.

10. Prove that if $x, y, z > 0$, then
    (a) $\frac{1}{\sqrt{xy}} + \frac{1}{\sqrt{yz}} + \frac{1}{\sqrt{zx}} \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.
    (b) $\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{x}{z} + \frac{z}{y} + \frac{y}{x}$.
    (c) $\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x\sqrt{\frac{y}{z}} + y\sqrt{\frac{z}{x}} + z\sqrt{\frac{x}{y}}$.
    (d) $x^4 + y^4 + z^4 \geq xyz(\sqrt{xy} + \sqrt{yz} + \sqrt{zx})$.

11. Prove that if $x, y > 0$, then
$$\frac{1}{x+y} \leq \frac{1}{4x} + \frac{1}{4y}.$$

12. Let $a, b, c \geq 0$ and $a+b+c \leq 3$. Prove that
$$\frac{a}{1+a^2} + \frac{b}{1+b^2} + \frac{c}{1+c^2} \leq \frac{3}{2} \leq \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}.$$

13. Prove the inequality $x^4 + y^4 + 8 \geq 8xy$ for positive real numbers x, y.

14. Prove that if a and b are positive real numbers, then
$$(1+\frac{a}{b})^n + (1+\frac{b}{a})^n \geq 2^{n+1}.$$

15. Prove that if $p, q > 0$ and $p+q = 1$, then
$$(p+\frac{1}{p})^2 + (q+\frac{1}{q})^2 \geq \frac{25}{2}.$$

16. Prove that if $x+y+z = 1$ then,
$$8(\frac{1}{2} - xy - yz - zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq 9.$$

17. Prove that if $a_1, a_2, ..., a_n$ are distinct positive real numbers and $a_1 + a_2 + ... + a_n = S$, then
$$\frac{S}{S-a_1} + \frac{S}{S-a_2} + ... + \frac{S}{S-a_n} > \frac{n^2}{n-1}.$$

18. Find the minimum value of
$$\frac{a_1}{1+a_2+a_3+...+a_{2009}} + \frac{a_2}{1+a_1+a_3...+a_{2009}} + ... + \frac{a_{2009}}{1+a_1+a_2+...+a_{2008}}$$
where $a_1, a_2, ..., a_{2009} > 0$ and $a_1 + a_2 + ... + a_{2009} = 1$.

19. Prove that for any $x \in \mathbb{R}$ we have $\frac{x^2}{1+x^4} \leq \frac{1}{2}$.

20. Let $x, y \geq 1$. Prove that $x\sqrt{y-1} + y\sqrt{x-1} \leq xy$.

21. Prove the inequality $\frac{x^2+2}{\sqrt{x^2+1}} \geq 2$ for any $x \in \mathbb{R}$.

22. Let $x > y > 0$ and $xy = 1$. Prove that $\frac{x^2+y^2}{x-y} > 2\sqrt{2}$.

23. Prove that if $x > y \geq 0$, then $x + \frac{4}{(x-y)(y+1)^2} \geq 3$.

24. Prove that if $a, b, c > 0$ then $\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c$.

25. Prove that if $a + b + c = 1$, then $a^2 + b^2 + c^2 \geq \frac{1}{3}$.

26. Prove that if $a + b + c = 3$, then $a^2 + b^2 + c^2 \geq 3 \geq ab + bc + ca$.

27. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that
$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \geq 1.$$

28. Prove that the inequality
$$a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} \geq ab + bc + ca$$
holds for any real numbers a, b, c.

29. Prove that if $x, y, z > 0$, then
$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{x}{y} + \frac{y}{z} + \frac{z}{x}.$$

30. Prove that if $x^3 + y^3 = 2$, then $x + y \leq 2$.

31. Let a, b, c be positive real numbers with $a^3 + b^3 + c^3 = 24$. Prove that $a + b + c \leq 6$.

32. Prove that if $a, b, c > 0$, then $\frac{a}{b+c+d} + \frac{b}{a+c+d} + \frac{c}{a+b+d} + \frac{d}{a+b+c} \geq \frac{4}{3}$.

33. Prove that if $a + b \geq 1$, then $a^4 + b^4 \geq \frac{1}{8}$.

34. Let $a, b, c > 0$. Prove that
$$\frac{a^3 - a + 2}{b+c} + \frac{b^3 - b + 2}{c+a} + \frac{c^3 - c + 2}{a+b} \geq 3.$$

35. Prove that for positive real numbers x, y, z we have
$$\frac{x}{x+2y+2z} + \frac{y}{y+2z+2x} + \frac{z}{z+2x+2y} \geq \frac{3}{5}.$$

36. Prove that if $a, b, c > 0$, then
$$\frac{a^4}{b^2c^2} + \frac{b^4}{c^2a^2} + \frac{c^4}{a^2b^2} \geq \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}.$$

37. Prove the inequality $2\sqrt{x+1} - 2\sqrt{x} < \frac{1}{\sqrt{x}} < 2\sqrt{x} - 2\sqrt{x-1}$ for $x \geq 1$.

38. Prove that if $a, b, c > 0$ then
$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}.$$

39. Prove that for $x, y > 0$,

$$\frac{1}{(1+\sqrt{x})^2} + \frac{1}{(1+\sqrt{y})^2} \geq \frac{2}{x+y+2}.$$

40. Prove that if $a, b, c > 0$ then

$$\frac{a^3+2}{3b+3c} + \frac{b^3+2}{3c+3a} + \frac{c^3+2}{3a+3b} \geq \frac{3}{2}.$$

41. Prove that if $a, b, c > 0$, then

$$1 + \frac{3}{ab+bc+ca} \geq \frac{6}{a+b+c}.$$

42. Solve the system in $\mathbb{R}^+$

$$\begin{aligned} a+b+c+d &= 12 \\ abcd &= 27 + ab + ac + ad + bc + bd + cd. \end{aligned}$$

43. Solve the system in the set of real numbers

$$\begin{aligned} \frac{4x^2}{4x^2+1} &= y, \\ \frac{4y^2}{4y^2+1} &= z, \\ \frac{4z^2}{4z^2+1} &= x. \end{aligned}$$

44. Solve the system where $x, y, z \in \mathbb{R}$

$$\begin{aligned} x + \frac{2}{x} &= 2y, \\ y + \frac{2}{y} &= 2z, \\ z + \frac{2}{z} &= 2x. \end{aligned}$$

45. Find the positive real numbers x, y, z, t satisfying the system

$$\begin{aligned} 16xyzt &= (x^2+y^2+z^2+t^2)(xyz+xyt+xzt+yzt), \\ 8 &= 2xy + 2zt + xz + xt + yz + yt. \end{aligned}$$

Chapter 2

CAUCHY-SCHWARZ INEQUALITY

Theorem 3 *(Cauchy-Schwarz)If x_i, y_i are real numbers, then*

$$(x_1^2 + x_2^2 + ... + x_n^2)(y_1^2 + y_2^2 + ... + y_n^2) \geq (x_1y_1 + x_2y_2 + ... + x_ny_n)^2$$

and equality holds if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = ... = \frac{x_n}{y_n}$.

Example 6 *Prove that $a^2+b^2+c^2 \geq \frac{(a+b+c)^2}{3}$ for any real numbers a, b, c.*

Solution: By Cauchy-Schwarz we have that

$$(1^2 + 1^2 + 1^2)(a^2 + b^2 + c^2) \geq (1 \cdot a + 1 \cdot b + 1 \cdot c)^2 = (a + b + c)^2.$$

Therefore, $a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3}$.

Example 7 *Let x, y, z be positive real numbers. Prove that*

$$\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+x} \geq \frac{x+y+z}{2}.$$

Solution: By Cauchy-Schwarz inequality we have

$$[(\sqrt{x+y})^2 + (\sqrt{y+z})^2 + (\sqrt{z+x})^2] \left((\frac{x}{\sqrt{x+y}})^2 + (\frac{y}{\sqrt{y+z}})^2 + (\frac{z}{\sqrt{z+x}})^2 \right) \geq$$

$$\left(\sqrt{x+y} \cdot \frac{x}{\sqrt{x+y}} + \sqrt{y+z} \cdot \frac{y}{\sqrt{y+z}} + \sqrt{z+x} \cdot \frac{z}{\sqrt{z+x}} \right)^2 = (x+y+z)^2$$

So, what we got is $(2x + 2y + 2z) \left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+x} \right) \geq (x + y + z)^2$. Therefore, $\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+x} \geq \frac{x+y+z}{2}$.

This is a very useful trick that can be applied to many problems. We can generalize this result as:

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + ... + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + ... + x_n)^2}{y_1 + y_2 + ... + y_n} \tag{2.1}$$

Example 8 *Let $a, b, c > 0$. Prove that* $\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} \geq \frac{a^2+b^2+c^2}{2}$.

Solution: We can write $\frac{a^3}{b+c} = \frac{a^4}{ab+ac}$ and similarly $\frac{b^3}{c+a} = \frac{b^4}{bc+ba}$ and $\frac{c^3}{a+b} = \frac{c^4}{ca+cb}$.
So by (2.1) we have

$$LHS = \frac{a^4}{ab+ac} + \frac{b^4}{bc+ba} + \frac{c^4}{ca+cb} \geq \frac{(a^2+b^2+c^2)^2}{2(ab+bc+ca)} \geq \frac{a^2+b^2+c^2}{2}$$

by (1.1) as desired.

2.1 EXERCISES

46. Prove that the inequality $x^2+y^2+z^2 \geq \frac{(x+2y+3z)^2}{14}$ holds for any $x, y, z \in \mathbb{R}$.

47. Prove that if $a, b, c > 0$, then
$$(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \geq 9.$$

48. Prove that if $x, y > 0$ and $x+3y = 1$ then $x^2+y^2 \geq \frac{1}{10}$.

49. a, b, c, x, y, z are real numbers and $a^2+b^2+c^2 = 16$, $x^2+y^2+z^2 = 25$ and $ax+by+cz = 20$. Compute $\frac{a+b+c}{x+y+z}$.

50. Prove that if $a, b > 0$, then $\sqrt{a}+\sqrt{b} \leq \sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}$.

51. Solve the system
$$x_1+x_2+...+x_k = 9$$
$$\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_k} = 1$$
where $x_i \in \mathbb{R}^+$.

52. Prove that if a, b, c are positive real numbers with $abc = 1$, then
$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a} \geq \frac{3}{2}.$$

53. Prove that if $x, y, z > 0$ and $x+y+z = 1$, then
$$8(\frac{1}{2}-xy-yz-zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right) \geq 9.$$

54. Prove that if $a, b, c > 0$ then
$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}.$$

55. Prove that if a, b, c, d are positive real numbers, then
$$\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

56. Let a, b, c, d be positive real numbers. Prove that
$$\sqrt{(a+c)(b+d)} \geq \sqrt{ab}+\sqrt{cd}.$$

57. Let $a > c > 0$ and $b > c > 0$. Prove that $\sqrt{c(a-c)}+\sqrt{c(b-c)} \leq \sqrt{ab}$.

58. Let $a, b, c > 0$ with $a + b + c = 3$. Prove that
$$\frac{a^2}{a+b+1} + \frac{b^2}{b+c+1} + \frac{c^2}{c+a+1} \geq 1.$$

59. Let $a, b, c > 0$ with $abc = 1$. Prove that
$$\frac{a^2}{a+b+1} + \frac{b^2}{b+c+1} + \frac{c^2}{c+a+1} \geq 1.$$

60. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that
$$\frac{a^3}{a+2b} + \frac{b^3}{b+2c} + \frac{c^3}{c+2a} \geq 1.$$

61. Let a, b, c, x, y, z be positive real numbers such that $x + y + z = 1$. Prove that
$$ax + by + cz + 2\sqrt{(xy + yz + zx)(ab + bc + ca)} \leq a + b + c.$$

62. Let a, b, c be positive real numbers. Prove that
$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}.$$

63. Prove that if $a, b, c, x, y, z > 0$, then
$$\frac{x^4}{a^3} + \frac{y^4}{b^3} + \frac{z^4}{c^3} \geq \frac{(x+y+z)^4}{(a+b+c)^3}.$$

64. Prove that if a, b, c are positive real numbers, then
$$\frac{a}{a+2b} + \frac{b}{b+2c} + \frac{c}{c+2a} \geq 1.$$

65. Prove that if a, b, c are positive real numbers, then
$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1.$$

66. Prove that if $a, b, c > 0$, then
$$\frac{a}{2a+b} + \frac{b}{2b+c} + \frac{c}{2c+a} \leq 1.$$

67. Let a, b, c, d be positive real numbers such that $(a^2 + b^2)^3 = c^2 + d^2$. Prove that
$$\frac{a^3}{c} + \frac{b^3}{d} \geq 1.$$

68. Let a, b be positive real numbers. Prove that

$$\frac{a^4+b^4}{a^3+b^3} \geq \frac{a^2+b^2}{a+b}.$$

69. Let $a, b, c > 0$ with $abc = 1$. Prove that

$$\frac{a^2+b^2+1}{a+b+1} + \frac{b^2+c^2+1}{b+c+1} + \frac{c^2+a^2+1}{c+a+1} \geq 3.$$

70. Prove that if $a, b, c, x, y, z > 0$ and $(a^2+b^2+c^2)^3 = x^2+y^2+z^2$, then

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq 1.$$

71. Prove that if a, b, c are positive real numbers, then

$$\frac{27a^2}{c} + \frac{(b+c)^2}{a} \geq 12b.$$

72. Let $a, b, c, d \geq 0$ with $ab + bc + cd + da = 1$. Prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

73. Let $a_1, a_2, ..., a_n; b_1, b_2, ..., b_n$ be positive real numbers such that $a_1 + a_2 + ... + a_n = b_1 + b_2 + ... + b_n$. Show that

$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + ... + \frac{a_n^2}{a_n+b_n} \geq \frac{a_1+a_2+...+a_n}{2}.$$

74. Let $a, b, c > 0$. Prove that

$$\frac{a}{b(b+c)^2} + \frac{b}{c(c+a)^2} + \frac{c}{a(a+b)^2} \geq \frac{9}{4(a^2+b^2+c^2)}.$$

75. Let $a, b, c > 0$ with $a + b + c = 3$. Prove that

$$\frac{a^2(b+1)}{a+b+ab} + \frac{b^2(c+1)}{b+c+bc} + \frac{c^2(a+1)}{c+a+ca} \geq 2.$$

Chapter 3

REARRANGEMENT INEQUALITY

Definition 6 *Let* $a_1 \leq a_2 \leq ... \leq a_n$ *and* $b_1 \leq b_2 \leq ... \leq b_n$*. The number* $A = a_1b_1 + a_2b_2 + ... + a_nb_n$ *is called ordered sum and the number* $B = a_1b_n + a_2b_{n-1} + ... + a_nb_1$ *is called reversed sum. And if* $x_1, x_2, ..., x_n$ *is a permutation of* $b_1, b_2, ..., b_n$ *then* $X = a_1x_1 + a_2x_2 + ... + a_nx_n$ *is called a mixed sum.*

Theorem 4 *Let* $a_1 \leq a_2 \leq ... \leq a_n$ *and* $b_1 \leq b_2 \leq ... \leq b_n$ *be given. For any mixed sum* X *we have* $B \leq X \leq A$*.*

Example 9 *Prove that for positive real numbers* a, b, c *the inequality* $a^2 + b^2 + c^2 \geq ab + bc + ca$ *holds.*

Solution: Since the inequality is symmetric, WLOG we can assume that $a \geq b \geq c$. So by Rearrangement inequality, being the ordered sum

$$a^2 + b^2 + c^2 = a \cdot a + b \cdot b + c \cdot c \geq a \cdot b + b \cdot c + c \cdot a.$$

Example 10 *Prove that if* a, b, c *are positive real numbers, then*

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq a + b + c.$$

Solution: WLOG, we can assume that $a \geq b \geq c$. Then $ab \geq ac \geq bc$ and $\frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a}$. By Rearrangement inequality we have,

$$LHS = ab \cdot \frac{1}{c} + ac \cdot \frac{1}{b} + bc \cdot \frac{1}{a} \geq ab \cdot \frac{1}{a} + ac \cdot \frac{1}{c} + bc \cdot \frac{1}{b} = a + b + c.$$

Example 11 *Prove that if a, b, c are positive real numbers, then*

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c.$$

Solution: By symmetry assume that $a \geq b \geq c$ then $a^3 \geq b^3 \geq c^3$ and $\frac{1}{bc} \geq \frac{1}{ca} \geq \frac{1}{ab}$. By Rearrangement inequality we have that

$$LHS = a^3 \cdot \frac{1}{bc} + b^3 \cdot \frac{1}{ca} + c^3 \cdot \frac{1}{ab} \geq a^3 \cdot \frac{1}{ca} + b^3 \cdot \frac{1}{ab} + c^3 \cdot \frac{1}{bc} = \frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b}$$

And again by Rearrangement inequality, being the reversed sum

$$a + b + c = a^2 \cdot \frac{1}{a} + b^2 \cdot \frac{1}{b} + c^2 \cdot \frac{1}{c} \leq a^2 \cdot \frac{1}{c} + b^2 \cdot \frac{1}{a} + c^2 \cdot \frac{1}{b}$$

Combining these two inequalities we get that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c.$$

3.1 EXERCISES

76. Let a, b, c be positive real numbers. Prove that
    (a) $a^3 + b^3 \geq ab(a + b)$
    (b) $a^5 + b^5 \geq ab(a^3 + b^3)$
    (c) $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$
    (d) $ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}$
    (e) $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c$
    (f) $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$
    (g) $abc(ab + bc + ca) \leq a^3b^2 + b^3c^2 + c^3a^2$
    (h) $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}$.

77. Prove that if a, b, c are positive real numbers, then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

78. Let $x_i \in \mathbb{R}^+$ and $y_1, y_2, ..., y_n$ be a permutation of $x_1, x_2, ..., x_n$. Prove that

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + ... + \frac{x_n^2}{y_n} \geq x_1 + x_2 + ... + x_n.$$

79. Let $x_1, x_2, ..., x_n$ be positive real numbers. Prove that

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + ... + \frac{x_n^2}{x_1} \geq x_1 + x_2 + ... + x_n.$$

80. Let $a_1, a_2, ..., a_n$ be distinct positive integers. Prove that

$$\frac{a_1}{1^2} + \frac{a_2}{2^2} + ... + \frac{a_n}{n^2} \geq \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n}.$$

81. Prove that if for any a, b, c we have

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \geq \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a^2}{c+a} + \frac{b^2}{a+b} + \frac{c^2}{b+c}$$

then $a = b = c$.

82. Let $a_1 \geq a_2 \geq a_3$ and $b_1 \geq b_2 \geq b_3$. Prove that

$$a_1b_1 + a_2b_2 + a_3b_3 \geq \frac{1}{3}(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)$$

Chapter 4

CHEBYSHEV'S INEQUALITY

Theorem 5 *Let* $a_1 \geq a_2 \geq ... \geq a_n$ *and* $b_1 \geq b_2 \geq ... \geq b_n$. *Then*

$$a_1b_1+a_2b_2+...+a_nb_n \geq \frac{1}{n}(a_1+a_2+...+a_n)(b_1+b_2+...+b_n) \geq a_1b_n+a_2b_{n-1}+...+a_nb_1.$$

Example 12 *Prove that* $a^2+b^2+c^2 \geq \frac{1}{3}(a+b+c)^2$.

Solution: By Chebyshev's inequality,

$$a^2+b^2+c^2 = a\cdot a+b\cdot b+c\cdot c \geq \frac{1}{3}(a+b+c)(a+b+c) = \frac{1}{3}(a+b+c)^2.$$

Example 13 *Let* $a,b,c>0$. *Prove that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}.$$

Solution: Due to symmetry, we can assume that $a \geq b \geq c$ then $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$. So by Chebyshev's inequality we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{1}{3}(a+b+c)(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}) \tag{4.1}$$

And by AM-HM we have

$$\frac{1}{3}(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}) \geq \frac{3}{b+c+c+a+a+b} = \frac{3}{2(a+b+c)} \tag{4.2}$$

Combining (4.1) and (4.2) we get,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}.$$

Example 14 *Prove that if $a, b, c > 0$ and $abc = 1$, then*

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{3}{2}.$$

Solution: WLOG, suppose that $a \geq b \geq c$. Then $a^2 \geq b^2 \geq c^2$ and $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$. So by Chebyshev's inequality we have,

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{1}{3}(a^2+b^2+c^2)(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b})$$

By Example 11 and (4.2) we have that

$$\frac{1}{3}(a^2+b^2+c^2)(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}) \geq \frac{(a+b+c)^2}{3} \cdot \frac{3}{2(a+b+c)} =$$

$$= \frac{a+b+c}{2} \geq \frac{\sqrt[3]{abc}}{2}$$

4.1 EXERCISES

83. Prove that if $a, b, c > 0$. then

    (a) $a^3 + b^3 \geq \frac{(a+b)(a^2+b^2)}{2}$

    (b) $\frac{a^{n+1}+b^{n+1}}{a^n+b^n} \geq \frac{a+b}{2}$

    (c) $a^4 + b^4 + c^4 \geq abc(a+b+c)$

    (d) $a^3 + b^3 + c^3 \geq \frac{(a+b+c)^3}{9}$

84. Prove that if $a, b, c > 0$, then

$$\frac{a}{a+2b+2c} + \frac{b}{b+2c+2a} + \frac{c}{c+2a+2b} \geq \frac{3}{5}.$$

85. Prove that if $a, b, c > 0$, then

$$\frac{a}{a+3b+3c} + \frac{b}{b+3c+3a} + \frac{c}{c+3a+3b} \geq \frac{3}{7}.$$

86. Prove that if $a, b, c > 0$ and $abc = 1$, then

$$\frac{a^2}{a+2b+2c} + \frac{b^2}{b+2c+2a} + \frac{c^2}{c+2a+2b} \geq \frac{3}{5}.$$

87. Prove that if $a, b, c > 0$, then

$$\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} \geq \frac{ab+bc+ca}{2}.$$

88. Prove that if $a, b, c > 0$ with $abc = 1$, then

$$\frac{a^3}{b+c+1} + \frac{b^3}{c+a+1} + \frac{c^3}{a+b+1} \geq 1.$$

89. Let $a, b, c, d \geq 0$ with $ab + bc + cd + da = 1$. Prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

90. Prove that if $a, b, c > 0$, then

$$\frac{b+c}{a+3b+3c} + \frac{c+a}{b+3c+3a} + \frac{a+b}{c+3a+3b} \leq \frac{6}{7}.$$

91. Prove that if $a, b, c > 0$, then

$$\frac{b+c}{a+2b+2c} + \frac{c+a}{b+2c+2a} + \frac{a+b}{c+2a+2b} \leq \frac{6}{5}.$$

92. Prove that if $a, b, c > 0$ and $abc = 1$ then

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} \geq 1.$$

93. Let x, y, z be positive real numbers with $xyz = 1$ and $a \geq 1$. Prove that

$$\frac{x^a}{y+z} + \frac{y^a}{z+x} + \frac{z^a}{x+y} \geq \frac{3}{2}.$$

94. Prove that if $a, b, c > 0$ with $a + b + c = 1$, then

$$\frac{a^3}{b^2+c^2} + \frac{b^3}{c^2+a^2} + \frac{c^3}{a^2+b^2} \geq \frac{1}{2}.$$

95. Prove that if $a, b, c > 0$ then

$$\frac{a^5}{b^3+c^3} + \frac{b^5}{c^3+a^3} + \frac{c^5}{a^3+b^3} \geq \frac{(a+b+c)^2}{6}.$$

96. Let $a, b, c > 0$. Prove that

$$\frac{a^3+b^3}{a^2+b^2} + \frac{b^3+c^3}{b^2+c^2} + \frac{c^3+a^3}{c^2+a^2} \geq a+b+c.$$

Chapter 5

MIXED PROBLEMS

97. Prove that if $a, b, c > 0$ and $abc = 1$, then

$$\frac{1}{a^3+b^3+1}+\frac{1}{b^3+c^3+1}+\frac{1}{c^3+a^3+1}\le 1.$$

98. Prove that if $a, b, c > 0$ and $abc = 1$, then

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\le 1.$$

99. Prove that if $x, y, z > 0$ then

$$\frac{xy}{x^2+xy+yz}+\frac{yz}{y^2+yz+zx}+\frac{zx}{z^2+zx+xy}\le 1.$$

100. Prove that if $x, y, z > 0$ then

$$\frac{xy}{3x^2+2y^2+z^2}+\frac{yz}{3y^2+2z^2+x^2}+\frac{zx}{3z^2+2x^2+y^2}\le \frac{1}{2}.$$

101. Prove that if $a, b, c > 0$ and $abc = 1$, then

$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\le 1.$$

102. Let a, b, c be real numbers such that $a^2+b^2+c^2=1$. Prove that

$$\frac{a^2}{1+2bc}+\frac{b^2}{1+2ca}+\frac{c^2}{1+2ab}\ge \frac{3}{5}.$$

103. (IMO95/2) Let a, b, and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\ge \frac{3}{2}.$$

104. Let $a, b, c > 0$ with $a + b + c = 1$. Prove that

$$\frac{a^3}{a^2+b^2} + \frac{b^3}{b^2+c^2} + \frac{c^3}{c^2+a^2} \geq \frac{1}{2}.$$

105. (IMO2000/2) Let a, b, c be positive real numbers with product 1. Prove that

$$(a - 1 + \frac{1}{b})(b - 1 + \frac{1}{c})(c - 1 + \frac{1}{a}) \leq 1.$$

106. Prove that if $a, b, c > 0$ and $abc = 1$, then

$$1 + \frac{3}{a+b+c} \geq \frac{6}{ab+bc+ca}.$$

107. Let a, b, c be positive real numbers such that $abc \leq 1$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c.$$

108. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{b+c}{\sqrt{a}} + \frac{c+a}{\sqrt{b}} + \frac{a+b}{\sqrt{c}} \geq \sqrt{a} + \sqrt{b} + \sqrt{c} + 3.$$

109. If $a, b, c \in (0, 1)$, then prove that

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$$

110. Prove that $\sqrt{a^2+(b-1)^2} + \sqrt{b^2+(c-1)^2} + \sqrt{c^2+(a-1)^2} \geq \frac{3\sqrt{2}}{2}$ for arbitrary real numbers a, b, c.

111. Prove that $3(a^2 - ab + b^2) \geq a^2 + ab + b^2$ for any real numbers a and b.

112. (Russia2002) Let x, y, z be positive real numbers with sum 3. Prove that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx.$$

113. Let a, b, c be positive real numbers. Prove that

$$\frac{ab}{a+b+2c} + \frac{bc}{b+c+2a} + \frac{ca}{c+a+2b} \leq \frac{a+b+c}{4}.$$

114. Prove that

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a+b+c}{3}$$

for positive real numbers a, b, c.

115. Prove that if $a, b, c > 0$, then

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq \frac{3(ab + bc + ca)}{a + b + c}.$$

116. (IMO2001/2)Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

117. Prove that if a, b, c are positive, then

$$\frac{a^3 + b^3}{a^2 + b^2} + \frac{b^3 + c^3}{b^2 + c^2} + \frac{c^3 + a^3}{c^2 + a^2} \geq a + b + c.$$

118. Prove that for positive real numbers a, b, c, d the inequality holds

$$\sqrt[3]{ab} + \sqrt[3]{cd} \leq \sqrt[3]{(a + c + b)(a + c + d)}.$$

119. Let x, y, z be positive reals with $x + y + z = 1$ and let $a = \sqrt{x^2 + xy + y^2}$, $b = \sqrt{y^2 + yz + z^2}, c = \sqrt{z^2 + zx + x^2}$. Prove that $ab + bc + ca \geq 1$.

120. Let $a, b, c > 0$. Prove that

$$\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a} \geq \frac{9}{a + b + c + \sqrt{3(ab + bc + ca)}}.$$

121. Let $a, b, c > 0$ with $abc = 1$. Prove that

$$\frac{a + b + 1}{a + b^2 + c^3} + \frac{b + c + 1}{b + c^2 + a^3} + \frac{c + a + 1}{c + a^2 + b^3} \leq \frac{(a + 1)(b + 1)(c + 1) + 1}{a + b + c}.$$

122. If $x, y > 0$ and $x^2 + y^3 \geq x^3 + y^4$. Prove that $x^3 + y^3 \leq 2$.

123. Let a, b, c be positive real numbers. Prove that

$$\frac{ab^3}{c} + \frac{bc^3}{a} + \frac{ca^3}{b} + 3abc \geq 2(a^2b + b^2c + c^2a).$$

124. Prove that if a, b, c are positive real numbers with $a + b + c = 1$, then

$$\frac{a^5}{a^4 + b^4} + \frac{b^5}{b^4 + c^4} + \frac{c^5}{c^4 + a^4} \geq \frac{1}{2}.$$

125. Let $a, b, c > 0$ with $abc = 1$. Prove that

$$\frac{a}{a^2 + 2} + \frac{b}{b^2 + 2} + \frac{c}{c^2 + 2} \leq 1.$$

126. (Mathematical Excalibur) Let $x, y, z > 1$. Prove that
$$\frac{x^4}{(y-1)^2} + \frac{y^4}{(z-1)^2} + \frac{z^4}{(x-1)^2} \geq 48.$$

127. (MMO/2009) Let $a, b, c > 0$ such that $ab + bc + ca = \frac{1}{3}$. Prove that
$$\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \geq \frac{1}{a + b + c}.$$

128. Prove that $\forall x, y, z \in \mathbb{R}^+$ we have:
$$\sum \frac{x^5}{x^3yz + yz^4} \geq \frac{3}{2}.$$

129. Let $x, y, z > 0$. Prove that
$$\frac{x^3}{y(x^2 + 2y^2)} + \frac{y^3}{z(y^2 + 2z^2)} + \frac{z^3}{x(z^2 + 2x^2)} \leq 1.$$

130. Prove that if $a, b, c > 0$, then
$$\frac{a}{a^2 + b + 1} + \frac{b}{b^2 + c + 1} + \frac{c}{c^2 + a + 1} \leq 1.$$

131. Prove that if $a, b, c > 0$ with $abc = 1$, then
$$\sum \frac{a^3 + b^3 + 1}{a^2 + b^2 + 1} \geq 3.$$

132. Prove that if $x, y, z > 0$, then
$$\sum \frac{x^2z^2 + y^4 + y^2z^2}{yz(xz + y^2 + yz)} \geq 3.$$

133. Let $a, b, c > 0$ with $(a + b)(b + c)(c + a) = 8$. Prove that
$$\frac{a + b + c}{3} \geq \sqrt[27]{\frac{a^3 + b^3 + c^3}{3}}.$$

134. Let x, y, z be positive reals. Prove that
$$\frac{yz}{2x^2 + yz} + \frac{zx}{2y^2 + zx} + \frac{xy}{2z^2 + xy} \geq 1.$$

135. Let x, y, z be positive reals. Prove that
$$\frac{x^2}{2x^2 + yz} + \frac{y^2}{2y^2 + zx} + \frac{z^2}{2z^2 + xy} \leq 1.$$

136. Let $a, b, x, y, z \in \mathbb{R}^+$. Prove that
$$\frac{x}{ay + bz} + \frac{y}{az + bx} + \frac{z}{ax + by} \geq \frac{3}{a + b}.$$

Chapter 6

PROBLEMS FROM OLYMPIADS

(From 'Inequalities Through Problems'-Hojoo Lee)

1 (**BMO 2005, Proposed by Serbia and Montenegro**) $(a, b, c > 0)$

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{4(a-b)^2}{a+b+c}$$

2 (**Romania 2005, Cezar Lupu**) $(a, b, c > 0)$

$$\frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

3 (**Romania 2005, Traian Tamaian**) $(a, b, c > 0)$

$$\frac{a}{b+2c+d} + \frac{b}{c+2d+a} + \frac{c}{d+2a+b} + \frac{d}{a+2b+c} \geq 1$$

4 (**Romania 2005, Cezar Lupu**) $\left(a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c},\ a, b, c > 0\right)$

$$a + b + c \geq \frac{3}{abc}$$

5 (**Romania 2005, Cezar Lupu**) $(1 = (a+b)(b+c)(c+a),\ a, b, c > 0)$

$$ab + bc + ca \geq \frac{3}{4}$$

6 (**Romania 2005, Robert Szasz**) $(a+b+c=3,\ a,b,c>0)$

$$a^2b^2c^2 \geq (3-2a)(3-2b)(3-2c)$$

7 (**Romania 2005**) $(abc \geq 1,\ a,b,c>0)$

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a} \leq 1$$

8 (**Romania 2005, Unused**) $(abc=1,\ a,b,c>0)$

$$\frac{a}{b^2(c+1)}+\frac{b}{c^2(a+1)}+\frac{c}{a^2(b+1)} \geq \frac{3}{2}$$

9 (**Romania 2005, Unused**) $(a+b+c \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ a,b,c>0)$

$$\frac{a^3c}{b(c+a)}+\frac{b^3a}{c(a+b)}+\frac{c^3b}{a(b+c)} \geq \frac{3}{2}$$

10 (**Romania 2005, Unused**) $(a+b+c=1,\ a,b,c>0)$

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}}$$

11 (**Romania 2005, Unused**) $(ab+bc+ca+2abc=1,\ a,b,c>0)$

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq \frac{3}{2}$$

12 (**Chzech and Solvak 2005**) $(abc=1,\ a,b,c>0)$

$$\frac{a}{(a+1)(b+1)}+\frac{b}{(b+1)(c+1)}+\frac{c}{(c+1)(a+1)} \geq \frac{3}{4}$$

13 (**Japan 2005**) $(a+b+c=1,\ a,b,c>0)$

$$a\left(1+b-c\right)^{\frac{1}{3}}+b\left(1+c-a\right)^{\frac{1}{3}}+c\left(1+a-b\right)^{\frac{1}{3}} \leq 1$$

14 (**Germany 2005**) $(a+b+c=1,\ a,b,c>0)$

$$2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{b}\right)\geq\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}$$

15 (**Vietnam 2005**) $(a,b,c>0)$

$$\left(\frac{a}{a+b}\right)^3+\left(\frac{b}{b+c}\right)^3+\left(\frac{c}{c+a}\right)^3\geq\frac{3}{8}$$

16 (**China 2005**) $(a+b+c=1,\ a,b,c>0)$

$$10(a^3+b^3+c^3)-9(a^5+b^5+c^5)\geq 1$$

17 (**China 2005**) $(abcd=1,\ a,b,c,d>0)$

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2}\geq 1$$

18 (**China 2005**) $(ab+bc+ca=\frac{1}{3},\ a,b,c\geq 0)$

$$\frac{1}{a^2-bc+1}+\frac{1}{b^2-ca+1}+\frac{1}{c^2-ab+1}\leq 3$$

19 (**Poland 2005**) $(0\leq a,b,c\leq 1)$

$$\frac{a}{bc+1}+\frac{b}{ca+1}+\frac{c}{ab+1}\leq 2$$

20 (**Poland 2005**) $(ab+bc+ca=3,\ a,b,c>0)$

$$a^3+b^3+c^3+6abc\geq 9$$

21 (**Baltic Way 2005**) $(abc=1,\ a,b,c>0)$

$$\frac{a}{a^2+2}+\frac{b}{b^2+2}+\frac{c}{c^2+2}\leq 1$$

22 (**Serbia and Montenegro 2005**) $(a, b, c > 0)$

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}(a+b+c)}$$

23 (**Serbia and Montenegro 2005**) $(a+b+c=3,\ a, b, c > 0)$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca$$

24 (**Bosnia and Hercegovina 2005**) $(a+b+c=1, a, b, c > 0)$

$$a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \leq \frac{1}{\sqrt{3}}$$

25 (**Iran 2005**) $(a, b, c > 0)$

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 \geq (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

26 (**Austria 2005**) $(a, b, c, d > 0)$

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \geq \frac{a+b+c+d}{abcd}$$

27 (**Moldova 2005**) $(a^4+b^4+c^4=3,\ a, b, c > 0)$

$$\frac{1}{4-ab} + \frac{1}{4-bc} + \frac{1}{4-ca} \leq 1$$

28 (**APMO 2005**) $(abc=8,\ a, b, c > 0)$

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}$$

29 (**IMO 2005**) $(xyz \geq 1, x, y, z > 0)$

$$\frac{x^5-x^2}{x^5+y^2+z^2} + \frac{y^5-y^2}{y^5+z^2+x^2} + \frac{z^5-z^2}{z^5+x^2+y^2} \geq 0$$

30 (**Poland 2004**) $(a+b+c=0, a,b,c\in\mathbb{R})$

$$b^2c^2+c^2a^2+a^2b^2+3\geq 6abc$$

31 (**Baltic Way 2004**) $(abc=1, a,b,c>0, n\in\mathbb{N})$

$$\frac{1}{a^n+b^n+1}+\frac{1}{b^n+c^n+1}+\frac{1}{c^n+a^n+1}\leq 1$$

32 (**Junior Balkan 2004**) $((x,y)\in\mathbb{R}^2-\{(0,0)\})$

$$\frac{2\sqrt{2}}{x^2+y^2}\geq\frac{x+y}{x^2-xy+y^2}$$

33 (**IMO Short List 2004**) $(ab+bc+ca=1,\ a,b,c>0)$

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a}\leq\frac{1}{abc}$$

34 (**APMO 2004**) $(a,b,c>0)$

$$(a^2+2)(b^2+2)(c^2+2)\geq 9(ab+bc+ca)$$

35 (**USA 2004**) $(a,b,c>0)$

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3)\geq(a+b+c)^3$$

36 (**Junior BMO 2003**) $(x,y,z>-1)$

$$\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq 2$$

37 (**USA 2003**) $(a,b,c>0)$

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\leq 8$$

38 (**Russia 2002**) $(x+y+z=3,\ x,y,z>0)$

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\geq xy+yz+zx$$

39 (**Latvia 2002**) $\left(\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1,\ a,b,c,d>0\right)$

$$abcd\geq 3$$

40 (**Albania 2002**) $(a,b,c>0)$

$$\frac{1+\sqrt{3}}{3\sqrt{3}}(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq a+b+c+\sqrt{a^2+b^2+c^2}$$

41 (**Belarus 2002**) $(a,b,c,d>0)$

$$\sqrt{(a+c)^2+(b+d)^2}+\frac{2|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}}\geq\sqrt{a^2+b^2}+\sqrt{c^2+d^2}\geq\sqrt{(a+c)^2+(b+d)^2}$$

42 (**Canada 2002**) $(a,b,c>0)$

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\geq a+b+c$$

43 (**Vietnam 2002, Dung Tran Nam**) $(a^2+b^2+c^2=9,\ a,b,c\in\mathbb{R})$

$$2(a+b+c)-abc\leq 10$$

44 (**Bosnia and Hercegovina 2002**) $(a^2+b^2+c^2=1,a,b,c\in\mathbb{R})$

$$\frac{a^2}{1+2bc}+\frac{b^2}{1+2ca}+\frac{c^2}{1+2ab}\geq\frac{3}{5}$$

45 (**Junior BMO 2002**) $(a,b,c>0)$

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)}\geq\frac{27}{2(a+b+c)^2}$$

46 (**Greece 2002**) $(a^2+b^2+c^2=1,\ a,b,c>0)$

$$\frac{a}{b^2+1}+\frac{b}{c^2+1}+\frac{c}{a^2+1}\geq\frac{3}{4}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$$

47 (**Greece 2002**) $(bc\neq 0, \frac{1-c^2}{bc}\geq 0,\ a,b,c\in\mathbb{R})$

$$10(a^2+b^2+c^2-bc^3)\geq 2ab+5ac$$

48 (**Taiwan 2002**) $\left(a,b,c,d\in\left(0,\frac{1}{2}\right]\right)$

$$\frac{abcd}{(1-a)(1-b)(1-c)(1-d)}\leq\frac{a^4+b^4+c^4+d^4}{(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4}$$

49 (**APMO 2002**) $(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1,\ x,y,z>0)$

$$\sqrt{x+yz}+\sqrt{y+zx}+\sqrt{z+xy}\geq\sqrt{xyz}+\sqrt{x}+\sqrt{y}+\sqrt{z}$$

50 (**Ireland 2001**) $(x+y=2,\ x,y\geq 0)$

$$x^2y^2(x^2+y^2)\leq 2.$$

51 (**BMO 2001**) $(a+b+c\geq abc,\ a,b,c\geq 0)$

$$a^2+b^2+c^2\geq\sqrt{3}abc$$

52 (**USA 2001**) $(a^2+b^2+c^2+abc=4,\ a,b,c\geq 0)$

$$0\leq ab+bc+ca-abc\leq 2$$

53 (**Columbia 2001**) $(x,y\in\mathbb{R})$

$$3(x+y+1)^2+1\geq 3xy$$

54 (**KMO Winter Program Test 2001**) $(a, b, c > 0)$

$$\sqrt{(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)} \geq abc + \sqrt[3]{(a^3 + abc)(b^3 + abc)(c^3 + abc)}$$

55 (**KMO Summer Program Test 2001**) $(a, b, c > 0)$

$$\sqrt{a^4 + b^4 + c^4} + \sqrt{a^2b^2 + b^2c^2 + c^2a^2} \geq \sqrt{a^3b + b^3c + c^3a} + \sqrt{ab^3 + bc^3 + ca^3}$$

56 (**IMO 2001**) $(a, b, c > 0)$

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

6.1 Years 1996 $\sim$ 2000

57 (**IMO 2000, Titu Andreescu**) $(abc = 1,\ a, b, c > 0)$

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$$

58 (**Czech and Slovakia 2000**) $(a, b > 0)$

$$\sqrt[3]{2(a + b)\left(\frac{1}{a} + \frac{1}{b}\right)} \geq \sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}}$$

59 (**Hong Kong 2000**) $(abc = 1,\ a, b, c > 0)$

$$\frac{1 + ab^2}{c^3} + \frac{1 + bc^2}{a^3} + \frac{1 + ca^2}{b^3} \geq \frac{18}{a^3 + b^3 + c^3}$$

60 (**Czech Republic 2000**) $(m, n \in N,\ x \in [0, 1])$

$$(1 - x^n)^m + (1 - (1 - x)^m)^n \geq 1$$

61 (**Macedonia 2000**) $(x, y, z > 0)$

$$x^2 + y^2 + z^2 \geq \sqrt{2}\,(xy + yz)$$

62 (**Russia 1999**) $(a, b, c > 0)$

$$\frac{a^2 + 2bc}{b^2 + c^2} + \frac{b^2 + 2ca}{c^2 + a^2} + \frac{c^2 + 2ab}{a^2 + b^2} > 3$$

63 (**Belarus 1999**) $(a^2 + b^2 + c^2 = 3,\ a, b, c > 0)$

$$\frac{1}{1 + ab} + \frac{1}{1 + bc} + \frac{1}{1 + ca} \geq \frac{3}{2}$$

64 (**Czech-Slovak Match 1999**) $(a, b, c > 0)$

$$\frac{a}{b + 2c} + \frac{b}{c + 2a} + \frac{c}{a + 2b} \geq 1$$

65 (**Moldova 1999**) $(a, b, c > 0)$

$$\frac{ab}{c(c + a)} + \frac{bc}{a(a + b)} + \frac{ca}{b(b + c)} \geq \frac{a}{c + a} + \frac{b}{b + a} + \frac{c}{c + b}$$

66 (**United Kingdom 1999**) $(p + q + r = 1,\ p, q, r > 0)$

$$7(pq + qr + rp) \leq 2 + 9pqr$$

67 (**Canada 1999**) $(x + y + z = 1,\ x, y, z \geq 0)$

$$x^2y + y^2z + z^2x \leq \frac{4}{27}$$

68 (**Proposed for 1999 USAMO**, [AB, pp.25]) $(x, y, z > 1)$

$$x^{x^2+2yz}y^{y^2+2zx}z^{z^2+2xy} \geq (xyz)^{xy+yz+zx}$$

69 (**Turkey, 1999**) $(c \geq b \geq a \geq 0)$

$$(a+3b)(b+4c)(c+2a) \geq 60abc$$

70 (**Macedonia 1999**) $(a^2+b^2+c^2=1,\ a,b,c>0)$

$$a+b+c+\frac{1}{abc} \geq 4\sqrt{3}$$

71 (**Poland 1999**) $(a+b+c=1,\ a,b,c>0)$

$$a^2+b^2+c^2+2\sqrt{3abc} \leq 1$$

72 (**Canda 1999**) $(x+y+z=1, x,y,z \geq 0)$

$$x^2y+y^2z+z^2x \leq \frac{4}{27}$$

73 (**Iran 1998**) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2,\ x,y,z>1\right)$

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$$

74 (**Belarus 1998**, I. Gorodnin) $(a,b,c>0)$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a+b}{b+c}+\frac{b+c}{a+b}+1$$

75 (**APMO 1998**) $(a,b,c>0)$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

76 (**Poland 1998**) $\left(a+b+c+d+e+f=1,\ ace+bdf \geq \frac{1}{108}\ a,b,c,d,e,f>0\right)$

$$abc+bcd+cde+def+efa+fab \leq \frac{1}{36}$$

77 (**Korea 1998**) $(x+y+z=xyz,\ x,y,z>0)$

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\le\frac{3}{2}$$

78 (**Hong Kong 1998**) $(a,b,c\ge 1)$

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}\le\sqrt{c(ab+1)}$$

79 (**IMO Short List 1998**) $(xyz=1,\ x,y,z>0)$

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)}\ge\frac{3}{4}$$

80 (**Belarus 1997**) $(a,x,y,z>0)$

$$\frac{a+y}{a+x}x+\frac{a+z}{a+x}y+\frac{a+x}{a+y}z\ge x+y+z\ge\frac{a+z}{a+z}x+\frac{a+x}{a+y}y+\frac{a+y}{a+z}z$$

81 (**Ireland 1997**) $(a+b+c\ge abc,\ a,b,c\ge 0)$

$$a^2+b^2+c^2\ge abc$$

82 (**Iran 1997**) $(x_1x_2x_3x_4=1,x_1,x_2,x_3,x_4>0)$

$$x_1^3+x_2^3+x_3^3+x_4^3\ge max\left(x_1+x_2+x_3+x_4,\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)$$

83 (**Hong Kong 1997**) $(x,y,z>0)$

$$\frac{3+\sqrt{3}}{9}\ge\frac{xyz(x+y+z+\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)(xy+yz+zx)}$$

84 (**Belarus 1997**) $(a,b,c>0)$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{a+b}{c+a}+\frac{b+c}{a+b}+\frac{c+a}{b+c}$$

85 (**Bulgaria 1997**) ($abc = 1,\ a, b, c > 0$)

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}$$

86 (**Romania 1997**) ($xyz = 1,\ x, y, z > 0$)

$$\frac{x^9+y^9}{x^6+x^3y^3+y^6} + \frac{y^9+z^9}{y^6+y^3z^3+z^6} + \frac{z^9+x^9}{z^6+z^3x^3+x^6} \geq 2$$

87 (**Romania 1997**) ($a, b, c > 0$)

$$\frac{a^2}{a^2+2bc} + \frac{b^2}{b^2+2ca} + \frac{c^2}{c^2+2ab} \geq 1 \geq \frac{bc}{a^2+2bc} + \frac{ca}{b^2+2ca} + \frac{ab}{c^2+2ab}$$

88 (**USA 1997**) ($a, b, c > 0$)

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}.$$

89 (**Japan 1997**) ($a, b, c > 0$)

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}$$

90 (**Estonia 1997**) ($x, y \in \mathbb{R}$)

$$x^2 + y^2 + 1 > x\sqrt{y^2+1} + y\sqrt{x^2+1}$$

91 (**APMC 1996**) ($x + y + z + t = 0, x^2 + y^2 + z^2 + t^2 = 1, x, y, z, t \in \mathbb{R}$)

$$-1 \leq xy + yz + zt + tx \leq 0$$

92 (**Spain 1996**) ($a, b, c > 0$)

$$a^2 + b^2 + c^2 - ab - bc - ca \geq 3(a-b)(b-c)$$

93 (**IMO Short List 1996**) $(abc = 1,\ a, b, c > 0)$

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \le 1$$

94 (**Poland 1996**) $\left(a + b + c = 1,\ a, b, c \ge -\frac{3}{4}\right)$

$$\frac{a}{a^2 + 1} + \frac{b}{b^2 + 1} + \frac{c}{c^2 + 1} \le \frac{9}{10}$$

95 (**Hungary 1996**) $(a + b = 1,\ a, b > 0)$

$$\frac{a^2}{a + 1} + \frac{b^2}{b + 1} \ge \frac{1}{3}$$

96 (**Vietnam 1996**) $(a, b, c \in \mathbb{R})$

$$(a + b)^4 + (b + c)^4 + (c + a)^4 \ge \frac{4}{7}\left(a^4 + b^4 + c^4\right)$$

97 (**Bearus 1996**) $(x + y + z = \sqrt{xyz},\ x, y, z > 0)$

$$xy + yz + zx \ge 9(x + y + z)$$

98 (**Iran 1996**) $(a, b, c > 0)$

$$(ab + bc + ca)\left(\frac{1}{(a + b)^2} + \frac{1}{(b + c)^2} + \frac{1}{(c + a)^2}\right) \ge \frac{9}{4}$$

99 (**Vietnam 1996**) $(2(ab + ac + ad + bc + bd + cd) + abc + bcd + cda + dab = 16, a, b, c, d \ge 0)$

$$a + b + c + d \ge \frac{2}{3}(ab + ac + ad + bc + bd + cd)$$

6.2 Years 1990 $\sim$ 1995

Any good idea can be stated in fifty words or less. S. M. Ulam

100 (**Baltic Way 1995**) $(a, b, c, d > 0)$

$$\frac{a+c}{a+b}+\frac{b+d}{b+c}+\frac{c+a}{c+d}+\frac{d+b}{d+a}\geq 4$$

101 (**Canda 1995**) $(a, b, c > 0)$

$$a^a b^b c^c \geq abc^{\frac{a+b+c}{3}}$$

102 (**IMO 1995, Nazar Agakhanov**) $(abc = 1,\ a, b, c > 0)$

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$$

103 (**Russia 1995**) $(x, y > 0)$

$$\frac{1}{xy}\geq\frac{x}{x^4+y^2}+\frac{y}{y^4+x^2}$$

104 (**Macedonia 1995**) $(a, b, c > 0)$

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\geq 2$$

105 (**APMC 1995**) $(m, n \in \mathbb{N},\ x, y > 0)$

$$(n-1)(m-1)(x^{n+m}+y^{n+m})+(n+m-1)(x^n y^m+x^m y^n)\geq nm(x^{n+m-1}y+xy^{n+m-1})$$

106 (**Hong Kong 1994**) $(xy + yz + zx = 1,\ x, y, z > 0)$

$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)\leq\frac{4\sqrt{3}}{9}$$

107 (**IMO Short List 1993**) $(a, b, c, d > 0)$

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}$$

108 (**APMC 1993**) $(a, b \geq 0)$

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \leq \frac{a+\sqrt[3]{a^2b}+\sqrt[3]{ab^2}+b}{4} \leq \frac{a+\sqrt{ab}+b}{3} \leq \sqrt{\left(\frac{\sqrt[3]{a^2}+\sqrt[3]{b^2}}{2}\right)^3}$$

109 (**Poland 1993**) $(x, y, u, v > 0)$

$$\frac{xy+xv+uy+uv}{x+y+u+v} \geq \frac{xy}{x+y} + \frac{uv}{u+v}$$

110 (**IMO Short List 1993**) $(a+b+c+d = 1,\ a, b, c, d > 0)$

$$abc + bcd + cda + dab \leq \frac{1}{27} + \frac{176}{27}abcd$$

111 (**Italy 1993**) $(0 \leq a, b, c \leq 1)$

$$a^2 + b^2 + c^2 \leq a^2b + b^2c + c^2a + 1$$

112 (**Poland 1992**) $(a, b, c \in \mathbb{R})$

$$(a+b-c)^2(b+c-a)^2(c+a-b)^2 \geq (a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

113 (**Vietnam 1991**) $(x \geq y \geq z > 0)$

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2$$

114 (**Poland 1991**) $(x^2 + y^2 + z^2 = 2,\ x, y, z \in \mathbb{R})$

$$x + y + z \leq 2 + xyz$$

115 (**Mongolia 1991**) $(a^2 + b^2 + c^2 = 2,\ a, b, c \in \mathbb{R})$

$$|a^3 + b^3 + c^3 - abc| \leq 2\sqrt{2}$$

116 (**IMO Short List 1990**) $(ab + bc + cd + da = 1,\ a, b, c, d > 0)$

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}$$

6.3 Supplementary Problems

117 (**Lithuania 1987**) $(x, y, z > 0)$

$$\frac{x^3}{x^2+xy+y^2}+\frac{y^3}{y^2+yz+z^2}+\frac{z^3}{z^2+zx+x^2}\geq\frac{x+y+z}{3}$$

118 (**Yugoslavia 1987**) $(a, b > 0)$

$$\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)\geq a\sqrt{b}+b\sqrt{a}$$

119 (**Yugoslavia 1984**) $(a, b, c, d > 0)$

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\geq 2$$

120 (**IMO 1984**) $(x+y+z=1,\ x, y, z \geq 0)$

$$0\leq xy+yz+zx-2xyz\leq\frac{7}{27}$$

121 (**USA 1980**) $(a, b, c \in [0,1])$

$$\frac{a}{b+c+1}+\frac{b}{c+a+1}+\frac{c}{a+b+1}+(1-a)(1-b)(1-c)\leq 1.$$

122 (**USA 1979**) $(x+y+z=1,\ x, y, z > 0)$

$$x^3+y^3+z^3+6xyz\geq\frac{1}{4}.$$

123 (**IMO 1974**) $(a, b, c, d > 0)$

$$1<\frac{a}{a+b+d}+\frac{b}{b+c+a}+\frac{c}{b+c+d}+\frac{d}{a+c+d}<2$$

124 (**IMO 1968**) $(x_1, x_2 > 0, y_1, y_2, z_1, z_2 \in \mathbb{R}, x_1y_1 > {z_1}^2, x_2y_2 > {z_2}^2)$

$$\frac{1}{x_1y_1 - {z_1}^2} + \frac{1}{x_2y_2 - {z_2}^2} \geq \frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2}$$

125 (**Nesbitt's inequality**) $(a, b, c > 0)$

$$\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{3}{2}$$

126 (**Polya's inequality**) $(a \neq b, a, b > 0)$

$$\frac{1}{3}\left(2\sqrt{ab} + \frac{a + b}{2}\right) \geq \frac{\ln b - \ln a}{b - a}$$

127 (**Klamkin's inequality**) $(-1 < x, y, z < 1)$

$$\frac{1}{(1 - x)(1 - y)(1 - z)} + \frac{1}{(1 + x)(1 + y)(1 + z)} \geq 2$$

128 (**Carlson's inequality**) $(a, b, c > 0)$

$$\sqrt[3]{\frac{(a + b)(b + c)(c + a)}{8}} \geq \sqrt{\frac{ab + bc + ca}{3}}$$

129 (**[ONI], Vasile Cirtoaje**) $(a, b, c > 0)$

$$\left(a + \frac{1}{b} - 1\right)\left(b + \frac{1}{c} - 1\right) + \left(b + \frac{1}{c} - 1\right)\left(c + \frac{1}{a} - 1\right) + \left(c + \frac{1}{a} - 1\right)\left(a + \frac{1}{b} - 1\right) \geq 3$$

130 (**[ONI], Vasile Cirtoaje**) $(a, b, c, d > 0)$

$$\frac{a - b}{b + c} + \frac{b - c}{c + d} + \frac{c - d}{d + a} + \frac{d - a}{a + b} \geq 0$$

131 (**Elemente der Mathematik, Problem 1207, Šefket Arslanagić**) $(x, y, z > 0)$

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq \frac{x+y+z}{\sqrt[3]{xyz}}$$

132 ($\sqrt{WURZEL}$, **Walther Janous**) $(x+y+z=1,\ x, y, z > 0)$

$$(1+x)(1+y)(1+z) \geq (1-x^2)^2+(1-y^2)^2+(1-z^2)^2$$

133 ($\sqrt{WURZEL}$, **Heinz-Jürgen Seiffert**) $(xy > 0, x, y \in \mathbb{R})$

$$\frac{2xy}{x+y}+\sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy}+\frac{x+y}{2}$$

134 ($\sqrt{WURZEL}$, **Šefket Arslanagić**) $(a, b, c > 0)$

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z} \geq \frac{(a+b+c)^3}{3\,(x+y+z)}$$

135 ($\sqrt{WURZEL}$, **Šefket Arslanagić**) $(abc = 1, a, b, c > 0)$

$$\frac{1}{a^2\,(b+c)}+\frac{1}{b^2\,(c+a)}+\frac{1}{c^2\,(a+b)} \geq \frac{3}{2}.$$

136 ($\sqrt{WURZEL}$, **Peter Starek, Donauwörth**) $(abc = 1, a, b, c > 0)$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3} \geq \frac{1}{2}\,(a+b)\,(c+a)\,(b+c)-1.$$

137 ($\sqrt{WURZEL}$, **Peter Starek, Donauwörth**) $(x+y+z = 3, x^2+y^2+z^2 = 7, x, y, z > 0)$

$$1+\frac{6}{xyz} \geq \frac{1}{3}\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)$$

138 (**$\sqrt{WURZEL}$, Šefket Arslanagić**) $(a, b, c > 0)$

$$\frac{a}{b+1} + \frac{b}{c+1} + \frac{c}{a+1} \geq \frac{3\,(a+b+c)}{a+b+c+3}.$$

139 $(a, b \geq 0)$

$$a^3(b+1) + b^3(a+1) \geq a^2(b+b^2) + b^2(a+a^2)$$

140 (**Latvia 1997**) $(n \in \mathbb{N},\ a, b, c > 0)$

$$\frac{1}{a+b} + \frac{1}{a+2b} + \cdots + \frac{1}{a+nb} < \frac{n}{\sqrt{a(a+nb)}}$$

141 (**[ONI], Gabriel Dospinescu, Mircea Lascu, Marian Tetiva**) $(a, b, c > 0)$

$$a^2 + b^2 + c^2 + 2abc + 3 \geq (1+a)(1+b)(1+c)$$

142 (**Gazeta Matematicã**) $(a, b, c > 0)$

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4} \geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}$$

143 (**C[1]2362, Mohammed Aassila**) $(a, b, c > 0)$

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \geq \frac{3}{1+abc}$$

144 (**C2580**) $(a, b, c > 0)$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{b+c}{a^2+bc} + \frac{c+a}{b^2+ca} + \frac{a+b}{c^2+ab}$$

145 (**C2581**) $(a, b, c > 0)$

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq a+b+c$$

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146 (**C2532**) $(a^2+b^2+c^2=1,\ a,b,c>0)$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq 3+\frac{2(a^3+b^3+c^3)}{abc}$$

147 (**C3032, Vasile Cirtoaje**) $(a^2+b^2+c^2=1,\ a,b,c>0)$

$$\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\leq\frac{9}{2}$$

148 (**C2645**) $(a,b,c>0)$

$$\frac{2(a^3+b^3+c^3)}{abc}+\frac{9(a+b+c)^2}{(a^2+b^2+c^2)}\geq 33$$

149 $(x,y\in\mathbb{R})$

$$-\frac{1}{2}\leq\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}\leq\frac{1}{2}$$

150 $(0<x,y<1)$

$$x^y+y^x>1$$

151 $(x,y,z>0)$

$$\sqrt[3]{xyz}+\frac{|x-y|+|y-z|+|z-x|}{3}\geq\frac{x+y+z}{3}$$

152 $(a,b,c,x,y,z>0)$

$$\sqrt[3]{(a+x)(b+y)(c+z)}\geq\sqrt[3]{abc}+\sqrt[3]{xyz}$$

153 $(x,y,z>0)$

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\leq 1$$

154 $(x+y+z=1,\ x,y,z>0)$

$$\frac{x}{\sqrt{1-x}}+\frac{y}{\sqrt{1-y}}+\frac{z}{\sqrt{1-z}}\geq\sqrt{\frac{3}{2}}$$

155 $(a,b,c\in\mathbb{R})$

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2}\geq\frac{3\sqrt{2}}{2}$$

156 $(a,b,c>0)$

$$\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}\geq\sqrt{a^2+ac+c^2}$$

157 $(xy+yz+zx=1,\ x,y,z>0)$

$$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}\geq\frac{2x(1-x^2)}{(1+x^2)^2}+\frac{2y(1-y^2)}{(1+y^2)^2}+\frac{2z(1-z^2)}{(1+z^2)^2}$$

158 $(x,y,z\geq 0)$

$$xyz\geq(y+z-x)(z+x-y)(x+y-z)$$

159 $(a,b,c>0)$

$$\sqrt{ab(a+b)}+\sqrt{bc(b+c)}+\sqrt{ca(c+a)}\geq\sqrt{4abc+(a+b)(b+c)(c+a)}$$

160 (**Darij Grinberg**) $(x,y,z\geq 0)$

$$\left(\sqrt{x\left(y+z\right)}+\sqrt{y\left(z+x\right)}+\sqrt{z\left(x+y\right)}\right)\cdot\sqrt{x+y+z}\geq 2\sqrt{\left(y+z\right)\left(z+x\right)\left(x+y\right)}.$$

161 (**Darij Grinberg**) $(x,y,z>0)$

$$\frac{\sqrt{y+z}}{x}+\frac{\sqrt{z+x}}{y}+\frac{\sqrt{x+y}}{z}\geq\frac{4\left(x+y+z\right)}{\sqrt{\left(y+z\right)\left(z+x\right)\left(x+y\right)}}.$$

162 (**Darij Grinberg**) $(a, b, c > 0)$

$$\frac{a^2\left(b+c\right)}{\left(b^2+c^2\right)\left(2a+b+c\right)}+\frac{b^2\left(c+a\right)}{\left(c^2+a^2\right)\left(2b+c+a\right)}+\frac{c^2\left(a+b\right)}{\left(a^2+b^2\right)\left(2c+a+b\right)}>\frac{2}{3}.$$

163 (**Darij Grinberg**) $(a, b, c > 0)$

$$\frac{a^2}{2a^2+\left(b+c\right)^2}+\frac{b^2}{2b^2+\left(c+a\right)^2}+\frac{c^2}{2c^2+\left(a+b\right)^2}<\frac{2}{3}.$$

164 (**Vasile Cirtoaje**) $(a, b, c \in \mathbb{R})$

$$(a^2+b^2+c^2)^2 \geq 3(a^3b+b^3c+c^3a)$$